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On the Convergence of the Boltzmann Equation for Semiconductors Toward the Energy Transport Model

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The diffusion limit of the Boltzmann equation of semiconductors is analyzed. The dominant collisions are the elastic collisions on one hand and the electron–electron collisions with the Pauli exclusion terms on the other hand. Under a nondegeneracy hypothesis on the distribution function, a lower bound of the entropy dissipation rate of the leading term of the Boltzmann kernel for semiconductors in terms of a distance to the space of Fermi–Dirac functions is proved. This estimate and a mean compactness lemma are used to prove the convergence of the solution of the Boltzmann equation to a solution of the energy transport model.

KEY WORDS: Boltzmann equation; semiconductor; diffusion; energy transport model; entropy dissipation rate.

1. INTRODUCTION

This paper is devoted to the proof of the convergence of the solution of the Boltzmann equation, for a degenerate semiconductor and with an arbitrary band structure, towards the solution of the Energy Transport model derived in ref. 4.

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The Energy Transport model (ET model) consists of a system of diffusion equations for the electronic density and energy. It improves the drift-diffusion model (DD model) in order to take into account the dependence of the mobility on the temperature and the thermal diffusion. It was first derived by Stratton[26] from the Boltzmann equation, by using phenomenologic closure relations Stratton's model is valid for a non-degenerate semiconductor (i.e., for which Pauli exclusion principle can be neglected) with a parabolic band structure. It has been widely used in numerical simulations,[1, 8, 13, 23, 25] but not much investigated from a mathematical point of view.

In ref. 4 an ET model is derived from the Boltzmann equation, by a Hilbert expansion, for a degenerate semiconductor with an arbitrary band structure. To this aim, the energy gain or loss of the electrons by the phonon collisions is assumed to be small, which yields that the phonon collision operator is the sum of an elastic operator and a small inelastic collision. Then, a diffusion limit of the Boltzmann equation is carried over, retaining as leading order terms the electron-'electron and elastic collisions.

In the present paper is proved the convergence of the solutions of the Boltzmann equation, to those of the ET model, in the framework of ref. 4.

Let us mention that in ref. 3 is performed the derivation of the ET model under a different assumption on the dominant collisions, which leads to the same model with different expressions of the diffusion coefficients.

The approach used here has been developed by Golse and Poupaud[21] for the DD model and is based on an entropy estimate and a mean compactness lemma. The mean compactness lemma used in the present study is proved in ref. 21 and is an adaptation of the result of Golse, Lions, Perthame and Sentis.[20] Here it is also necessary to study the link between the conservative and entropic variables, which was immediate in ref. 21.

The entropy estimate stated in the present paper is similar to the one established by Desvillettes in ref. 15. However, in the framework of ref. 15 (the theory of rarefied gazes) the energy is a parabolic function of the kinetic variable, which is not true in the present study. Due to this non parabolic structure, the proof presented here is different.

Similarly as in the work of ref. 15, the entropy estimate presented here is stated in L^2 and relies on the assumption that the solution of the scaled Boltzmann equation is bounded from below and above, uniformly with respect to the time, position, kinetic variable and to the small parameter of the asymptotic development, see Theorem 1 and Remark 2.2. This assumption is very strong. Indeed, it seems possible to establish it for a fixed value of the small parameter, in a time interval near zero, but the measure of this interval might tend to zero as the small parameter tends to zero. This assumption is also used, in this paper, in the study of the link between the conservative and entropic variables. It is also very close to the assumption of non-degeneracy of the diffusion matrix in the ET model, used in refs. 11 and 12 to prove the existence of solutions of the latter. One way to avoid it could be to look for an estimate in a weighted L^2 space.

This paper is organized as follows. In Section 2 are given the setting of the problem, the assumptions and the result. In Section 3 is stated the entropy estimate and Section 4 is devoted to the mean compactness lemma and to the link between the conservative and entropic variables. The proof of the convergence is finished is Section 5.

2. SETTING OF THE PROBLEM AND MAIN RESULT

In this paper is considered the same framework as in ref. 4. The starting point is the Boltzmann equation for a degenerate semiconductor (i.e., Pauli exclusion principle is taken into account) with an arbitrary band structure. Electron–electron collisions as well as impurity and phonon collisions are incorporated:

$$\frac{\partial f}{\partial t}+\frac{1}{\hbar}\nabla_k \varepsilon(k)\cdot\nabla_x f+\frac{q}{\hbar}\nabla_x V\cdot\nabla_k f=Q_e(f)+Q_i(f)+Q_{ph}(f) \tag{2.1}$$

In this picture, the electrons are described by their distribution function $f(t, x, k)$, where t is the time variable, x is the position variable lying in a bounded domain Ω of $\mathbb{R}^3$ and k is the wave vector lying in the first Brillouin zone B. (The first Brillouin zone is the elementary cell of the dual lattice L^* and is identified to the torus $\mathbb{R}^3/L^*$). The dynamics of electrons is described by the equations

$$\frac{dx}{dt}=v(k)=\frac{1}{\hbar}\nabla_k\varepsilon(k), \qquad \frac{d\hbar k}{dt}=q\,\nabla_x V$$

where $\varepsilon(k)$ is the energy band, V is the electrostatic potential, $\hbar$ is the reduced Planck constant and q the elementary charge. The electrostatic potential is in general deduced from the distribution function through the Coulomb interaction, but for the sake of simplicity, we shall assume here that it is given and does not depend on time.

The electron–electron collision operator reads

$$Q_e(f)(k)=\int_{B^3}\{f'f'_1(1-f)(1-f_1)-ff_1(1-f')(1-f'_1)\}$$
$$\times\,\delta_\varepsilon\,\delta_p\,\Phi_e(k,k',k_1,k'_1)\,dk_1\,dk'\,dk'_1 \tag{2.2}$$

where δ_ε stands for $\delta(\varepsilon(k)+\varepsilon(k_1)-\varepsilon(k')-\varepsilon(k'_1))$ and models the conservation of the kinetic energy during a collision, δ_p stands for $\sum_{g\in L^*}\delta(k+k_1-k'-k'_1+g)$, and models a periodized conservation of the impulsion: k, k_1, k' and k'_1 lie in the Brillouin zone and $k+k_1-k'-k'_1=g$ where $g\in L^*$. Equation (2.2) has a meaning using the coarea formula,[18] see Eq. (2.16). Finally, in formula (2.2), Φ_e is the cross section, f, f_1, f', f'_1 stand respectively for $f(k)$, $f(k_1)$, $f(k')$, $f(k'_1)$, The terms $(1-f)$, $(1-f_1)$,... express the Pauli exclusion principle and lead to the natural bound $0\leqslant f\leqslant 1$.

Electron-impurity collisions are elastic and are modelled by

$$Q_i(f)(k)=\int_B (f'-f)\,\delta(\varepsilon(k')-\varepsilon(k))\,\Phi_i(k,k')\,dk' \tag{2.3}$$

where Φ_i is a cross section.

The electron–phonon collision operator reads

$$\begin{aligned}Q_{ph}(f)(k)=\int_B &\Phi_{ph}(k,k')\\ &\times\{[(N_{ph}+1)\,\delta(\varepsilon-\varepsilon'+\varepsilon_{ph})+N_{ph}\,\delta(\varepsilon-\varepsilon'-\varepsilon_{ph})]\,f'(1-f)\\ &-[(N_{ph}+1)\,\delta(\varepsilon'-\varepsilon+\varepsilon_{ph})+N_{ph}\,\delta(\varepsilon'-\varepsilon-\varepsilon_{ph})]\,f(1-f')\}\,dk'\end{aligned} \tag{2.4}$$

where ε_{ph} and N_{ph} are respectively the phonon energy and occupation number, and Φ_{ph} is a cross section.

In ref. 4 is derived an ET model from Eq. (2.1) under an assumption on the collision operators and after a rescaling of the equation. It is assumed that the typical energy of a phonon, $\varepsilon_{ph,0}$, is small compared with the typical kinetic energy of an electron, ε_0. The latter is used as energy unit to rescale Eq. (2.1). A small parameter α is introduced: $\alpha^2=\varepsilon_{ph,0}/\varepsilon_0$. Expanding formally the scaled Q_{ph} in powers of α^2 yields

$$Q_{ph}(f)=Q_{ph,0}(f)+\alpha^2 Q_1^\alpha$$

where $Q_{ph,0}(f)$ is an elastic operator and Q_1^α an inelastic correction. It is assumed that $Q_{ph,0}$, Q_i and Q_e have the same order of magnitude and an operator Q_0 is introduced modelling all elastic collisions:

$$Q_0(f)=Q_{ph,0}(f)+Q_i(f)=\int_B (f'-f)\,\delta(\varepsilon(k')-\varepsilon(k))\,\Phi_0(k,k')\,dk'$$

where Φ_0 is the corresponding cross-section. In ref. 4, the leading order operators are Q_e and Q_0.

The inelastic electron–phonon operator, Q_1^α, will not be investigated in this paper. It will give the energy relaxation term in the ET model, see Eq. (2.31). For the sake of simplicity, we require in this study a uniform bound of the operator, Assumption 4. This operator with arbitrary band structure and the Pauli principle has not been investigated up to now. We refer to ref. 24 for a study of this operator for a parabolic band diagram and without the Pauli principle.

In ref. 4, a diffusion limit of the Boltzmann equation is performed: the macroscopic time and length scale are related to the kinetic ones according to $t_m = \alpha^2 t_k$ and $x_m = \alpha x_k$, (t_m, x_m) being the macroscopic scale, (t_k, x_k) the kinetic one. Then, the rescaled Boltzmann equation reads

$$\frac{\partial f^\alpha}{\partial t} + \frac{1}{\alpha} (\nabla_k \varepsilon(k) \cdot \nabla_x + \nabla_x V \cdot \nabla_k)\, f^\alpha = \frac{1}{\alpha^2} (Q_e(f^\alpha) + Q_0(f^\alpha)) + Q_1^\alpha(f^\alpha) \tag{2.5}$$

This equation is supplemented with the initial condition

$$\forall x \in \Omega, k \in B, \qquad f^\alpha(0, x, k) = f^\alpha_{in}(x, k) \tag{2.6}$$

and the boundary conditions are described by a scattering operator relating the incoming and outgoing part of f, as in ref. 14: $\forall t \in \mathbb{R}_+$, $x \in \partial\Omega$, $k \in B_-(x)$,

$$f^\alpha(t, x, k) = \int_{B_+(x)} R(k' \to k)\, \delta(\varepsilon(k) - \varepsilon(k'))\, f^\alpha(t, x, k')\, dk' \tag{2.7}$$

where $B_\pm(x) = \{k \in B, \pm \nabla_k \varepsilon(k) \cdot \nu(x) > 0\}$, $\nu(x)$ is the outward unit normal at $x \in \partial\Omega$, and $R(k' \to k)$ is a given cross section. The delta function in Eq. (2.7) expresses that the underlying microscopic dynamics is elastic. Therefore, reflections occur with conservation of the total (kinetic) energy and mass.

2.1. Assumptions

We shall in this subsection precise the assumptions that will be needed in the sequel.

Assumption 1. The energy band

- The function ε: $B \to \mathbb{R}_+$ belongs to $C^2(\bar{B})$, has at most a finite number of critical points and even (with respect to k). Denoting $k = (k^1, k^2, k^3)$

we assume that the functions 1, $\partial\varepsilon/\partial k^1$, $\partial\varepsilon/\partial k^2$, $\partial\varepsilon/\partial k^3$ are linearly independent. Moreover, we assume that ε satisfies:

$$\exists C, \xi > 0, \qquad \forall \omega \in S^3, \gamma > 0, \qquad \left|\left\{k \in B, \left|\begin{pmatrix}\nabla_k \varepsilon(k)\\ 1\end{pmatrix} \cdot \omega\right| \leqslant \gamma\right\}\right| \leqslant C\gamma^{\xi} \tag{2.8}$$

where $|\,|$ denotes (and will denote from now on) the Lebesgue measure on B.

- Let us define for $(k, k_1, g) \in B \times B \times L^*$ the function

$$\tilde{\varepsilon}(k, k_1, g)(k') = \varepsilon(k') + \varepsilon(k + k_1 + g - k') - \varepsilon(k) - \varepsilon(k_1) \tag{2.9}$$

Its domain of definition is the set

$$B_{k, k_1, g} = \{k' \in B, k + k_1 + g - k' \in B\} \tag{2.10}$$

We assume that for any $(k, k_1, g) \in B \times B \times L^*$, the function $\tilde{\varepsilon}(k, k_1, g)$ has at most a finite number of critical points.

Assumption 1 expresses the non degeneracy of the band diagram. It has a real three dimensional structure. This is the case for band diagrams of real materials.

Assumption 2. Cross sections, microreversibility

We assume that Φ_e and Φ_0 satisfy the following identities

$$\forall (k, k', k_1, k_1') \in B^4, \qquad \Phi_e(k, k', k_1, k_1') = \Phi_e(k', k, k_1', k_1) = \Phi_e(k_1, k_1', k, k'), \tag{2.11}$$

$$\Phi_0(k, k') = \Phi_0(k', k)$$

With these assumptions, formulas (2.2) and (2.3) can be understood thanks to the Co-area formula (see ref. 18). Indeed, for $e \in \varepsilon(B)$, the manifold $\varepsilon^{-1}(e) = \{k \in B, \varepsilon(k) = e\}$ has at most a finite number of singularities thanks to Assumption 1. Denote by $dS_e(k)$ its Euclidean surface element and by $N(e)$ the density of states of energy e:

$$N(e) = \int_{k \in \varepsilon^{-1}(e)} dN_e(k), \qquad dN_e(k) = \frac{dS_e(k)}{|\nabla \varepsilon(k)|} \tag{2.12}$$

The elastic collision operator Q_0 reads

$$Q_0(f)(k) = \int_{k' \in \varepsilon^{-1}(\varepsilon(k))} \Phi_0(k, k')(f' - f)\, dN_{\varepsilon(k)}(k') \tag{2.13}$$

In the same way, we consider for all $(k, k_1, g) \in B \times B \times L^*$ the manifold

$$\tilde{\varepsilon}^{-1}(k, k_1, g)(0) = \{k' \in B, k + k_1 + g - k' \in B, \tilde{\varepsilon}(k, k_1, g)(k') = 0\} \tag{2.14}$$

where $\tilde{\varepsilon}$ is defined in (2.9). This manifold has also at most a finite number of singularities thanks to Assumption 1. We denote by $d\tilde{S}(k, k_1, g)(k')$ its Euclidian surface element and by $\tilde{N}(k, k_1, g)$ the following density of states:

$$\begin{aligned} \tilde{N}(k, k_1, g) &= \int_{k' \in \tilde{\varepsilon}^{-1}(k, k_1, g)(0)} d\tilde{N}_{k, k_1, g}(k'), \\ d\tilde{N}_{k, k_1, g}(k') &= \frac{d\tilde{S}(k, k_1, g)(k')}{|\nabla_{k'} \tilde{\varepsilon}(k, k_1, g)(k')|} \end{aligned} \tag{2.15}$$

Let

$$\mathscr{P}_{k, k_1} = \{g \in L^*, \tilde{\varepsilon}^{-1}(k, k_1, g)(0) \neq \varnothing\}$$

which is finite since B is bounded. Then, Q_e can be written thanks to the Co-area formula under the form

$$\begin{aligned} Q_e(f)(k) = \int_{k_1 \in B} \sum_{g \in \mathscr{P}_{k, k_1}} dk_1 \int_{k' \in \tilde{\varepsilon}^{-1}(k, k_1, g)(0)} & d\tilde{N}_{k, k_1, g}(k') \\ \times \Phi_e(k, k', k_1, k + k_1 + g - k')(& f' f(k + k_1 + g - k')(1 - f) \\ \times (1 - f_1) - f f_1 (1 - f')(1 - & f(k + k_1 + g - k'))) \end{aligned} \tag{2.16}$$

We shall also use the notation

$$\bar{N}(k, k_1) = \sum_{g \in \mathscr{P}_{k, k_1}} \tilde{N}(k, k_1, g) \tag{2.17}$$

Assumption 3. Amplitude of cross sections

- There exist two constants $c_0, C_0 > 0$ such that for a.e. $k, k' \in B^2$ verifying $\varepsilon(k) = \varepsilon(k')$,

$$c_0 \leqslant \Phi_0(k, k')\, N(\varepsilon(k)) \leqslant C_0 \tag{2.18}$$

- There exist two constants $c_e, C_e > 0$ such that when $k, k_1 \in B^2$, and $k' \in \bigcup_{g \in \mathscr{P}_{k,k_1}} \tilde{\varepsilon}^{-1}(k, k_1, g)(0)$,

$$c_e \leqslant \Phi_e(k, k', k_1, k + k_1 - k') \, \tilde{N}(k, k_1, 0) \tag{2.19}$$

$$\bar{N}(k, k_1) \left\{ \sum_{g \in \mathscr{P}_{k,k_1}} \Phi_e(k, k', k_1, k + k_1 + g - k') \right\} \leqslant C_e \tag{2.20}$$

Assumption 4. Inelastic operators

We shall not give an explicit form for the inelastic operator Q_1^α. We assume however that $Q_1^\alpha(f) = Q_1^0(f) + \alpha Q_{1,1}^\alpha(f)$ and both Q_1^0 and $Q_{1,1}^\alpha$ are bounded operators of $L^2(B)$ (uniformly in α for the second one) such that for any centered Fermi–Dirac function $F(k) = \exp(a + c\varepsilon(k))/(1 + \exp(a + c\varepsilon(k)))$ (where a and c are real numbers),

$$\int_B Q_1^0(F) \, dk = 0 \tag{2.21}$$

Assumption 5. Natural bounds for the initial condition

The function f_{in}^α lies in $L^\infty(\Omega \times B)$ and satisfies for a.e. $(x, v) \in \Omega \times B$:

$$0 \leqslant f_{in}^\alpha(x, v) \leqslant 1 \tag{2.22}$$

Assumption 5 is natural for densities constrained to verify Pauli's exclusion principle, which is the case in a degenerate semiconductor.

Assumption 6. Regularity of the electric field

The function V belongs to $C^2(\bar{\Omega})$.

Assumption 7. Reflection operator on the boundary

The open set Ω of $\mathbb{R}^3$ is regular (C^2) and connected. The cross section $R(k' \to k)$ is a nonnegative measure satisfying the following identities.

For all $(x, k) \in \partial\Omega \times B$ such that $k \in B_+(x)$,

$$|\nabla_k \varepsilon(k) \cdot \nu(x)| = \int_{B_-(x)} |\nabla_k \varepsilon(k') \cdot \nu(x)| \, R(k \to k') \, \delta(\varepsilon(k) - \varepsilon(k')) \, dk' \tag{2.23}$$

and for all $(x, k, k') \in \partial\Omega \times B \times B$ such that $k \in B_+(x)$ and $k' \in B_-(x)$,

$$|\nabla_k \varepsilon(k') \cdot \nu(x)| \, R(k \to k') = |\nabla_k \varepsilon(-k) \cdot \nu(x)| \, R(-k' \to -k) \tag{2.24}$$

Equation (2.23) means that the boundary restitutes all the impinging electrons without altering their energy. Indeed a simple computation proves that (2.23) leads to

$$\int_{B_-(x)} G(\varepsilon(k))\, |\nabla_k \varepsilon(k) \cdot \nu(x)|\, f(t, x, k)\, dk$$

$$= \int_{B_+(x)} G(\varepsilon(k))\, |\nabla_k \varepsilon(k) \cdot \nu(x)|\, f(t, x, k)\, dk \tag{2.25}$$

for all f satisfying (2.7) and all functions G. Equation (2.24) is a reciprocity relation resulting from the time reversibility of the microscopic dynamics, see refs. 14 or 6 and references therein.

We refer to ref. 4 for a detailed physical interpretation of this framework, as well as for a discussion of the relevant bibliography.

2.2. The Result

Let us first introduce the following definition.

Definition 1. We say that f^α is a weak solution of (2.5)–(2.7) under Assumptions 1 to 7 if $f^\alpha \in C^0([0, T], L^2(\Omega \times B))$, f^α admits a trace $f^\alpha_\pm$ on the set $\{(t, x, k) \in [0, T] \times \partial\Omega \times B, k \in B_\pm(x)\}$, and for all test function $\theta \in \mathscr{D}([0, T[\times \bar{\Omega} \times \bar{B})$, the following weak formulation is verified,

$$\int_{\Omega \times B} f^\alpha_{in}(x, k)\, \theta(0, x, k)\, dx\, dk$$

$$- \int_0^T \int_{\Omega \times B} f^\alpha \left[\frac{\partial \theta}{\partial t} + \frac{1}{\alpha} (\nabla_k \varepsilon(k) \cdot \nabla_x \theta + \nabla_x V \cdot \nabla_k \theta) \right] dx\, dk\, dt$$

$$= \int_0^T \int_{\Omega \times B} \theta(t, x, k) \left[\frac{1}{\alpha^2} (Q_e(f^\alpha) + Q_0(f^\alpha)) + Q_1^\alpha(f^\alpha) \right] dx\, dk\, dt$$

$$- \mathscr{B}(f^\alpha, \theta) \tag{2.26}$$

where the boundary term is (thanks to (2.7) together with (2.23))

$$\mathscr{B}(f^\alpha, \theta) = \int_0^T \int_{\partial\Omega} \int_{k \in B_+(x)} \int_{k \in B_-(x)} |\nabla \varepsilon(k') \cdot \nu(x)|\, f^\alpha_+(x, k, t)\, R(k \to k')$$

$$\times \delta(\varepsilon(k) - \varepsilon(k'))[\theta(x, k, t) - \theta(x, k', t)]\, dk\, dk'\, d\sigma(x)\, dt \tag{2.27}$$

The aim of this paper is to prove the following result:

Theorem 1. Let f^α be a weak solution to the rescaled problem (2.5)–(2.7) under Assumptions 1 to 7 in the sense of Definition 1. Assume that there exists $\beta > 0$ such that for almost every $(\alpha, t, x, k) \in]0, 1] \times [0, T] \times \Omega \times B$,

$$\beta \leqslant f^\alpha(t, x, k) \leqslant 1 - \beta \tag{2.28}$$

Then, up to extraction of a subsequence, $f^\alpha(t, x, k)$ converges in $L^2([0, T] \times \Omega \times B)$ strong when α tends to 0 to a centered Fermi–Dirac equilibrium $F^0(t, x, k)$. Its moments are

$$\rho^0(t, x) = \int_B F^0(t, x, k)\, dk, \qquad W^0(t, x) = \int_B F^0(t, x, k)\, \varepsilon(k)\, dk \tag{2.29}$$

They solve in the the weak sense the following Energy Transport model,

$$\frac{\partial \rho^0}{\partial t} + \nabla_x \cdot J^0 = 0 \tag{2.30}$$

$$\frac{\partial W^0}{\partial t} + \nabla_x \cdot J^0_W - \nabla_x V \cdot J^0 = \int_B Q^0_1(F^0)\, \varepsilon(k)\, dk \tag{2.31}$$

with the homogeneous boundary conditions

$$J^0 \cdot \nu(x) = J^0_W \cdot \nu(x) = 0 \qquad \forall x \in \partial\Omega \tag{2.32}$$

The current density and the energy current density are given by the formulae

$$J^0 = \int_B r^0\, \nabla_k \varepsilon(k)\, dk \tag{2.33}$$

$$J^0_W = \int_B r^0 \varepsilon(k)\, \nabla_k \varepsilon(k)\, dk \tag{2.34}$$

where $r^0 \in L^2([0, T] \times \Omega \times B)$ satisfies the following equation:

$$(\nabla_k \varepsilon(k) \cdot \nabla_x + \nabla_x V \cdot \nabla_k)\, F^0 = (D^1 Q_e(F^0) + Q_0)(r^0) \tag{2.35}$$

The initial condition for ρ^0 and W^0 are the limit as α tends to zero of $\int f^\alpha_{in}\, dk$ and $\int f^\alpha_{in}\, \varepsilon\, dk$.

Remark 2.1. The limit equations listed in the above theorem are identical to the Energy Transport model derived in ref. 4. The formulation of Theorem 1 is more tractable for the present study. In order to introduce the diffusion coefficients of ref. 4, we first notice that in the above equations we can replace r^0 by

$$f_1(t, x, k) = r^0(t, x, k) + a(t, x) + b(t, x)\, \varepsilon(k)$$

since the last two terms give a zero contribution when they are multiplied by $\varepsilon\, \nabla\varepsilon$ or $\nabla\varepsilon$ and integrated over the whole Brillouin zone B. Now we can choose $a(t, x)$ and $b(t, x)$ in such a way that the integrals of f_1 and $f_1 \varepsilon$ over the Brillouin zone vanish. Then, we recover the situation of ref. 4 since f_1 satisfies

$$(\nabla_k \varepsilon(k) \cdot \nabla_x + \nabla_x V \cdot \nabla_k)\, F^0 = (D^1 Q_e(F^0) + Q_0)(f_1)$$

Indeed, writing

$$F^0 = \frac{1}{1 + \exp((\varepsilon - \mu)/T)}$$

leads to

$$f_1(x, k, t) = \left[\nabla_x \cdot \frac{\mu}{T} - \frac{\nabla_x V}{T} \right] \cdot \Psi_1 + \nabla_x \left(\frac{1}{T} \right) \cdot \Psi_2$$

where Ψ_1 and Ψ_2 are the unique solutions of

$$(D^1 Q_e(F^0) + Q_0)(\Psi_1) = \nabla_k \varepsilon F^0 (1 - F^0)$$

$$(D^1 Q_e(F^0) + Q_0)(\Psi_2) = \varepsilon\, \nabla_k \varepsilon F^0 (1 - F^0)$$

such that

$$\int_B \Psi_i\, dk = \int_B \varepsilon(k)\, \Psi_i(k)\, dk = 0 \qquad i = 1, 2$$

After some computations we end up with the following formulae

$$J^0 = D_{11} \left[\nabla_x \cdot \frac{\mu}{T} - \frac{\nabla_x V}{T} \right] + D_{12} \frac{\nabla_x T}{T^2}$$

$$J^0_W = D_{21} \left[\nabla_x \cdot \frac{\mu}{T} - \frac{\nabla_x V}{T} \right] + D_{22} \frac{\nabla_x T}{T^2}$$

where the matrices D_{ij} are given by

$$D_{1j}=\int_B \nabla\varepsilon(k)\otimes \Psi_j(k)\,dk, \qquad D_{2j}=\int_B \varepsilon(k)\,\nabla\varepsilon(k)\otimes \Psi_j(k)\,dk$$

Remark 2.2. We do not prove here rigorously the existence of the weak solutions f^α of the Boltzmann equation. We explain however briefly how this can be done.

Notice first that the estimate $0\leqslant f^\alpha\leqslant 1$ remains valid for all times if it is satisfied at time $t=0$ (maximum principle[17, 21]). It is then possible to prove that the map

$$f\to \mathcal{Q}_\alpha(f)=\frac{Q_e(f)+Q_0(f)}{\alpha^2}+Q_1^\alpha(f)$$

is continuous from the set $\{0\leqslant f\leqslant 1\}$ endowed with the $L^2(B)$ norm, on $L^2(B)$.

A fixed point argument then shows the existence for our weak problem. For the treatment of the boundary term, we refer to refs. 22 and 7.

Finally, the bound $0\leqslant f^\alpha\leqslant 1$ implies that $f^\alpha\in C^0(\mathbb{R}^+, L^p(\Omega\times B))$ for all $p<+\infty$.

The assumption $\beta\leqslant f\leqslant 1-\beta$ is very strong and difficult to prove, especially when one is looking for global (in time) solutions. Indeed, it should not be difficult to prove by continuity arguments that if the initial data satisfy this bound (and are sufficiently regular), then the solution of the Boltzmann equation satisfies the same bound with β replaced by $\beta/2$ in a time interval near zero. The problem is that the measure of this interval might tend to zero as α goes to zero.

On the other hand, Degond, Génieys and Jüngel have shown the existence of solutions of the Energy Transport model[9–12] under the hypothesis that the diffusion matrices do not degenerate. This hypothesis is not fulfilled for example if the temperature approaches zero.

Note that under this non-degeneracy assumption, one could hope to prove the assumption $\beta\leqslant f^\alpha\leqslant 1-\beta$ on a (small) time interval independent of α.

The proof of Theorem 1 will be done in several steps. In Section 3, we prove that the distance in L^2 of f^α towards the set of centered Fermi–Dirac distribution functions tends to zero. The main tool here is an entropy dissipation estimate, in the spirit of the works of refs. 15 and 28. Then, averaging lemmas are used in Section 4 in order to prove the strong convergence of the moments of f^α. The strong convergence of f^α itself towards a centered Fermi–Dirac function is then obtained as a corollary. Finally, the passage to the limit leading to Eqs. (2.30)–(2.35) is performed in Section 5, following the moment approach of the previous works.[2, 21]

Remark. In the sequel, the following properties of symmetry deduced from the co-area formula (see ref. 18) will be used systematically:

(i) For any measurable $f\colon B^2 \to \mathbb{R}$ such that the integrals below converge,

$$\int_{k \in B} \int_{k' \in \varepsilon^{-1}(\varepsilon(k))} f(k, k')\, dN_{\varepsilon(k)}(k')\, dk = \int_{k \in B} \int_{k' \in \varepsilon^{-1}(\varepsilon(k))} f(k', k)\, dN_{\varepsilon(k)}(k')\, dk \tag{2.36}$$

(ii) For any measurable $f\colon B^2 \to \mathbb{R}$ such that the integrals below converge,

$$\begin{aligned}
&\int_{B^2} dk\, dk_1 \sum_{g \in \mathscr{P}_{k, k_1}} \int_{k' \in \tilde{\varepsilon}^{-1}(k, k_1, g)(0)} d\tilde{N}_{k, k_1, g}(k')\, f(k, k', k_1, k+k_1+g-k') \\
&= \int_{B^2} dk\, dk_1 \sum_{g \in \mathscr{P}_{k, k_1}} \int_{k' \in \tilde{\varepsilon}^{-1}(k, k_1, g)(0)} d\tilde{N}_{k, k_1, g}(k')\, f(k', k, k+k_1+g-k', k_1) \\
&= \int_{B^2} dk\, dk_1 \sum_{g \in \mathscr{P}_{k, k_1}} \int_{k' \in \tilde{\varepsilon}^{-1}(k, k_1, g)(0)} d\tilde{N}_{k, k_1, g}(k')\, f(k_1, k+k_1+g-k', k, k')
\end{aligned} \tag{2.37}$$

(iii) For any $k, k', k_1, k'_1 \in B$ such that $\varepsilon(k) + \varepsilon(k_1) = \varepsilon(k') + \varepsilon(k'_1)$ and $k + k_1 - k' - k'_1 \in L^*$, one has

$$\bar{N}(k, k_1) = \bar{N}(k', k'_1) \tag{2.38}$$

3. ENTROPY DISSIPATION RATE AND DEPARTURE FROM THE EQUILIBRIUM

Let us denote by $\mathscr{F}$ and $\mathscr{F}_c$ the respective sets of Fermi–Dirac and centered Fermi–Dirac functions:

$$\mathscr{F} = \left\{ \frac{\exp(a + b \cdot k + c\varepsilon(k))}{1 + \exp(a + b \cdot k + c\varepsilon(k))},\ a, c \in \mathbb{R}, b \in \mathbb{R}^3 \right\} \tag{3.1}$$

$$\mathscr{F}_c = \left\{ \frac{\exp(a + c\varepsilon(k))}{1 + \exp(a + c\varepsilon(k))},\ a, c \in \mathbb{R} \right\} \tag{3.2}$$

We also introduce the entropy dissipations relative to the collision operators Q_0 and Q_e

$$E_{Q_e}(f)=\int_B Q_e(f)\,H(f)\,dk, \qquad E_{Q_0}(f)=\int_B Q_0(f)\,H(f)\,dk \tag{3.3}$$

and the global entropy dissipation

$$E_g(f)=E_{Q_e}(f)+E_{Q_0}(f) \tag{3.4}$$

Here, H is the function defined by

$$H(y)=\ln\left(\frac{y}{1-y}\right), \qquad \text{for} \quad 0<y<1 \tag{3.5}$$

The main result of this section is the following estimate:

Proposition 3.1. For any $\beta>0$, there exists a constant $C_\beta>0$ such that for all measurable functions $f\colon B\to\mathbb{R}$ satisfying $\beta\leqslant f\leqslant 1-\beta$ a.e.,

$$-E_g(f)\geqslant C_\beta \inf_{F\in\mathscr{F}_c} \|f-F\|^2_{L^2(B)} \tag{3.6}$$

The proof is done in the spirit of ref. 15 and is decomposed into several lemmas.

Lemma 3.2. For any $\beta>0$, there exists a constant $C_{1,\beta}>0$ such that for all measurable functions $f\colon B\to\mathbb{R}$ satisfying $\beta\leqslant f\leqslant 1-\beta$ a.e.,

$$-E_{Q_0}(f)\geqslant C_{1,\beta} \inf_{U\in L^\infty(\mathbb{R})} \int_B |H(f)(k)-U(\varepsilon(k))|^2\,dk \tag{3.7}$$

Proof of Lemma 3.2. From now on, we shall use the notation $\varepsilon=\varepsilon(k)$, $\varepsilon'=\varepsilon(k')$. Thanks to the properties of symmetry of Φ_0 (see Assumption 2), we can write

$$\begin{aligned}-E_{Q_0}(f)&=\tfrac{1}{2}\int_{B^2}\Phi_0(k,k')\,\delta(\varepsilon'-\varepsilon)(f-f')(H(f)-H(f'))\,dk\,dk'\\ &=\tfrac{1}{2}\int_{B^2}\Phi_0(k,k')\,\delta(\varepsilon'-\varepsilon)\,f'(1-f)\,\lambda(H(f)-H(f'))\,dk\,dk' \end{aligned}\tag{3.8}$$

where

$$\lambda(x) = x(e^x - 1) \tag{3.9}$$

Recalling that $\beta \leqslant f \leqslant 1-\beta$, there exists a constant $K_\beta > 0$ such that for all $(k, k') \in B^2$,

$$\lambda(H(f) - H(f')) \geqslant K_\beta (H(f) - H(f'))^2 \tag{3.10}$$

Hence, using the Co-area formula, we obtain

$$\begin{aligned} -E_{Q_0}(f) &\geqslant \tfrac{1}{2}\beta(1-\beta)\, K_\beta \int_{B^2} \Phi_0(k,k')\, \delta(\varepsilon' - \varepsilon)(H(f) - H(f'))^2 \, dk\, dk' \\ &\geqslant \tfrac{1}{2}\beta(1-\beta)\, K_\beta \int_B \int_{k' \in \varepsilon^{-1}(\varepsilon(k))} \Phi_0(k,k')(H(f) - H(f'))^2 \, dN_{\varepsilon(k)}(k') \end{aligned} \tag{3.11}$$

Under Assumption 2 and Jensen's inequality, we get

$$\begin{aligned} -E_{Q_0}(f) &\geqslant \frac{1}{2}\beta(1-\beta)\, K_\beta c_0 \int_B \left| \int_{k' \in \varepsilon^{-1}(\varepsilon(k))} (H(f) - H(f')) \frac{dN_{\varepsilon(k)}(k')}{N(\varepsilon(k))} \right|^2 dk \\ &\geqslant C_{1,\beta} \int_B \left| H(f) - \int_{k' \in \varepsilon^{-1}(\varepsilon(k))} H(f') \frac{dN_{\varepsilon(k)}(k')}{N(\varepsilon(k))} \right|^2 dk \\ &\geqslant C_{1,\beta} \inf_{U \in L^\infty(\mathbb{R})} \int_B |H(f(k)) - U(\varepsilon(k))|^2 \, dk \end{aligned} \tag{3.12}$$

Lemma 3.3. For any $\beta > 0$, there exists a constant $C_{2,\beta} > 0$ such that for all measurable function $f: B \to \mathbb{R}$ satisfying $\beta \leqslant f \leqslant 1-\beta$ a.e.,

$$\begin{aligned} -E_{Q_e}(f) \geqslant C_{2,\beta} &\inf_{T \in L^\infty(B \times \mathbb{R})} \\ &\times \int_{B^2} |H(f) + H(f_1) - T(k + k_1, \varepsilon(k) + \varepsilon(k_1))|^2 \, dk\, dk_1 \end{aligned} \tag{3.13}$$

Proof of Lemma 3.3. Thanks to the symmetry properties of Φ_e (see Assumption 2), we can write (using the notation $d^4k = dk\, dk_1\, dk'\, dk'_1$):

$$
\begin{aligned}
-E_{Q_e}(f) &= \frac{1}{4} \int_{B^4} \Phi_e\, \delta_\varepsilon\, \delta_p (f f_1 (1-f')(1-f'_1) - f' f'_1 (1-f)(1-f_1)) \\
&\quad \times (H(f) + H(f_1) - H(f') - H(f'_1))\, d^4k \\
&= \frac{1}{4} \int_{B^4} \Phi_e\, \delta_\varepsilon\, \delta_p f' f'_1 (1-f)(1-f_1) \\
&\quad \times \lambda (H(f) + H(f_1) - H(f') - H(f'_1))\, d^4k \\
&\geqslant C_\beta \int_{B^4} \Phi_e\, \delta_\varepsilon\, \delta_p (H(f) + H(f_1) - H(f') - H(f'_1))^2\, d^4k \\
&\geqslant C_\beta \int_{B^2} dk\, dk_1 \sum_{g \in \mathscr{P}_{k, k_1}} \int_{\tilde{\varepsilon}(k, k_1, g)^{-1}(0)} d\tilde{N}_{k, k_1, g}(k') \\
&\quad \times \Phi_e(k, k', k_1, k + k_1 + g - k') \\
&\quad \times (H(f) + H(f_1) - H(f') - H(f(k + k_1 + g - k')))^2
\end{aligned} \tag{3.14}
$$

The right hand side of the above identity is a sum of nonnegative terms. Therefore, using only one term ($g=0$), Assumption 3 and Jensen's inequality yields

$$
\begin{aligned}
-E_{Q_e}(f) &\geqslant C_{2,\beta} \int_{B^2} \Bigg| \int_{\tilde{\varepsilon}(k, k_1, 0)^{-1}(0)} (H(f) + H(f_1) \\
&\quad - H(f') - H(f(k + k_1 - k'))) \frac{d\tilde{N}_{k, k_1, 0}(k')}{\tilde{N}(k, k_1, 0)} \Bigg|^2 dk\, dk_1 \\
&\geqslant C_{2,\beta} \int_{B^2} \Bigg| H(f) + H(f_1) - \int_{\tilde{\varepsilon}(k, k_1, 0)^{-1}(0)} H(f') \\
&\quad + H(f(k + k_1 - k')) \frac{d\tilde{N}_{k, k_1, 0}(k')}{\tilde{N}(k, k_1, 0)} \Bigg|^2 dk\, dk_1 \\
&\geqslant C_{2,\beta} \inf_{T \in L^\infty(B \times \mathbb{R})} \int_{B^2} \Big| H(f) + H(f_1) \\
&\quad - T(k + k_1, \varepsilon(k) + \varepsilon(k_1)) \Big|^2 dk\, dk_1
\end{aligned} \tag{3.15}
$$

Lemma 3.4. There exists a constant $C_3 > 0$ such that for all measurable function $f\colon B \to \,]0, 1[$ satisfying

$$H(f) \in L^2(B) \tag{3.16}$$

the following estimate holds:

$$\inf_{T \in L^\infty(B \times \mathbb{R})} \int_{B^2} |H(f) + H(f_1) - T(k + k_1, \varepsilon(k) + \varepsilon(k_1))|^2 \, dk \, dk_1 \tag{3.17}$$

$$\geqslant C_3 \inf_{m \in M} \int_B |H(f) - m|^2 \, dk \tag{3.18}$$

where M is the set

$$M = \{a + b \cdot k + c\varepsilon(k),\ a,\ c \in \mathbb{R},\ b \in \mathbb{R}^3\} \tag{3.19}$$

Proof of Lemma 3.4. Let $\mathscr{B}$ be the set of functions of $L^2(B^2)$ depending only on $k + k_1$ and $\varepsilon(k) + \varepsilon(k_1)$ and introduce the following linear operator:

$$\begin{aligned} L\colon L^2(B)/M &\to L^2(B^2)/\mathscr{B} \\ t(k) &\mapsto Lt(k, k_1) = t(k) + t(k_1) \end{aligned} \tag{3.20}$$

Inequality (3.18) is satisfied if and only if the map L is open. Consequently, we shall prove that L is continuous, one to one and has a closed range and then apply the open mapping theorem.

(I) *L is continuous.* Since $m(k) + m(k_1)$ is in $\mathscr{B}$ whenever m is in M, and since B is bounded, there exists a positive constant C such that

$$\inf_{T \in \mathscr{B}} \int_{B^2} |t(k) + t(k_1) - T(k, k_1)|^2 \, dk \, dk_1$$

$$\leqslant \inf_{m \in M} \int_{B^2} |t(k) - m(k) + t(k_1) - m(k_1)|^2 \, dk \, dk_1$$

$$\leqslant C \inf_{m \in M} \int_B |t(k) - m(k)|^2 \, dk \tag{3.21}$$

(II) *The range of L is closed.* Let t_n be a sequence in $L^2(B)/M$ and u in $L^2(B^2)/\mathscr{B}$ such that Lt_n tends to u in $L^2(B^2)/\mathscr{B}$. Let us prove that u is in the range of L. First, there exists a sequence s_n in $L^2(B)$, a sequence $T_n(k + k_1, \varepsilon(k) + \varepsilon(k_1))$ in $L^2(B^2)$ (as a function of k and k_1) and g in $L^2(B^2)$ such that t_n is the natural projection of s_n on $L^2(B)/M$, u is the natural projection of g on $L^2(B^2)/\mathscr{B}$ and

$$s_n(k) + s_n(k_1) + T_n(k + k_1, \varepsilon(k) + \varepsilon(k_1)) \to g(k, k_1) \tag{3.22}$$

in $L^2(B^2)$. Writing $k=(k^1,k^2,k^3)$ and $k_1=(k_1^1,k_1^2,k_1^3)$, we introduce the differential operators, for $(i,j)\in\{1,2,3\}^2$,

$$\tilde{\nabla}_{ij}=\left(\left(\frac{\partial\varepsilon}{\partial k^i}\right)(k)-\left(\frac{\partial\varepsilon}{\partial k^i}\right)(k_1)\right)\left(\frac{\partial}{\partial k^j}-\frac{\partial}{\partial k_1^j}\right)$$
$$-\left(\left(\frac{\partial\varepsilon}{\partial k^j}\right)(k)-\left(\frac{\partial\varepsilon}{\partial k^j}\right)(k_1)\right)\left(\frac{\partial}{\partial k^i}-\frac{\partial}{\partial k_1^i}\right) \tag{3.23}$$

which enjoy the following property:

For $(i,j)\in\{1,2,3\}^2$,

$$\tilde{\nabla}_{ij}(T_n(k+k_1,\varepsilon(k)+\varepsilon(k_1)))=0 \tag{3.24}$$

Therefore, for $(i,j)\in\{1,2,3\}^2$,

$$\tilde{\nabla}_{ij}(s_n(k)+s_n(k_1))\to\tilde{\nabla}_{ij}g(k,k_1)\qquad\text{in } H^{-1}(B^2) \tag{3.25}$$

So far the proof is a rewriting of the previous proof[15] for the Boltzmann equation. The only difference is that the energy band is not parabolic. In ref. 15, the proof goes on by applying a certain differential operator Rio (3.25) and for which many terms (involving the third derivative of the band diagram) vanish. This cannot be done in our case because the band diagram is not parabolic and consequently its third derivative does not vanish.

We propose an alternative proof relying on the use of test functions. According to Assumption 1, for all $(i,j)\in\{1,2,3\}^2$ such that $i\neq j$ there exists a test function $\phi^{ij}\in H_0^1(B)$ such that

$$\langle 1,\phi^{ij}(k_1)\rangle=0 \tag{3.26}$$

$$\left\langle\left(\frac{\partial\varepsilon}{\partial k^i}\right)(k_1),\phi^{ij}(k_1)\right\rangle=1 \tag{3.27}$$

$$\left\langle\left(\frac{\partial\varepsilon}{\partial k^j}\right)(k_1),\phi^{ij}(k_1)\right\rangle=0 \tag{3.28}$$

where $\langle\ ,\ \rangle$ is the H^{-1},H_0^1 duality product (cf. for example ref. 5, p. 41, Lemma 3.2). Taking the duality product of (3.25) with $\phi^{ij}(k^1)$ (for $i\neq j$), we obtain the convergence in $H^{-1}(B)$ of

$$A_n^{ij}(k)=\frac{\partial s_n}{\partial k^j}(k)+a_n^{ij}\frac{\partial\varepsilon}{\partial k^i}(k)-b_n^{ij}\frac{\partial\varepsilon}{\partial k^j}(k)-c_n^{ij} \tag{3.29}$$

where

$$a_n^{ij} = \left\langle \left(\frac{\partial s_n}{\partial k^j} \right)(k_1), \phi^{ij}(k_1) \right\rangle, \qquad b_n^{ij} = \left\langle \left(\frac{\partial s_n}{\partial k^i} \right)(k_1), \phi^{ij}(k_1) \right\rangle \tag{3.30}$$

and

$$c_n^{ij} = \left\langle \left(\frac{\partial \varepsilon}{\partial k^i} \right)(k_1) \left(\frac{\partial s_n}{\partial k^j} \right)(k_1) - \left(\frac{\partial \varepsilon}{\partial k^j} \right)(k_1) \left(\frac{\partial s_n}{\partial k^i} \right)(k_1), \phi^{ij}(k_1) \right\rangle \tag{3.31}$$

Replacing in (3.25) $(\partial S_n/\partial k_j)(k)$ by the value deduced from (3.29), we get the convergence in $H^{-1}(B \times B)$ of

$$\begin{aligned}
&(b_n^{ij} - b_n^{ji}) \left\{ \frac{\partial \varepsilon}{\partial k^i}(k) \frac{\partial \varepsilon}{\partial k^j}(k) - \frac{\partial \varepsilon}{\partial k^i}(k_1) \frac{\partial \varepsilon}{\partial k^j}(k) \right. \\
&\qquad \left. - \frac{\partial \varepsilon}{\partial k^i}(k) \frac{\partial \varepsilon}{\partial k^j}(k_1) + \frac{\partial \varepsilon}{\partial k^i}(k_1) \frac{\partial \varepsilon}{\partial k^j}(k_1) \right\} \\
&\qquad + a_n^{ij} \left\{ - \left(\frac{\partial \varepsilon}{\partial k^i} \right)^2 (k) + 2 \frac{\partial \varepsilon}{\partial k^i}(k) \frac{\partial \varepsilon}{\partial k^i}(k_1) - \left(\frac{\partial \varepsilon}{\partial k^i} \right)^2 (k_1) \right\} \\
&\qquad + a_n^{ji} \left\{ \left(\frac{\partial \varepsilon}{\partial k^j} \right)^2 (k) - 2 \frac{\partial \varepsilon}{\partial k^j}(k) \frac{\partial \varepsilon}{\partial k^j}(k_1) + \left(\frac{\partial \varepsilon}{\partial k^j} \right)^2 (k_1) \right\}
\end{aligned} \tag{3.32}$$

Then, testing this convergence against the functions $\phi^{ab}(k)\, \phi^{cd}(k_1)$, with $ab, cd = ij$ or ji, we get the convergence of $(b_n^{ij} - b_n^{ji})$, a_n^{ij} and a_n^{ji}. Therefore, $b_n^{ij} = b_n + \tilde{b}_n^{ij}$ where $\tilde{b}_n^{ij}$ is bounded. Consequently, after the extraction of a subsequence, (3.29) can be rewritten

$$\tilde{A}_n^{ij}(k) = \frac{\partial s_n}{\partial k^j}(k) - b_n \frac{\partial \varepsilon}{\partial k^j}(k) - c_n \tag{3.33}$$

where $\tilde{A}_n^{ij}(k)$ converges in $H^{-1}(B)$. Hence, there exists a sequence of real numbers d_n such that

$$s_n(k) - b_n \varepsilon(k) - c_n \cdot k + d_n \tag{3.34}$$

converges in $L^2(B)$ (where $c_n = ((c_n)_1, (c_n)_2, (c_n)_3)$). Therefore, the range of L is closed.

(III) *L is one to one.* The previous arguments are still valid here. Let $s \in L^2(B)$, and assume that there exists a function T (in $L^2(B^2)$ as function of k, k_1) such that

$$s(k) + s(k_1) = T(k + k_1, \varepsilon(k) + \varepsilon(k_1)) \tag{3.35}$$

Then, using again for $(i, j) \in \{1, 2, 3\}^2$ the operators $\tilde{\nabla}_{ij}$ defined in Eq. (3.23), it can be proved that there exists $a, d \in \mathbb{R}$ and $c \in \mathbb{R}^3$ such that

$$s(k) = -d + c \cdot k + b\varepsilon(k) \in M \tag{3.36}$$

The proof is a rewriting the proof of closedness of L in which the subscript n is removed and the expressions "bounded" or "converges in H^{-1}" replaced by "equal to zero."

We now come to the

Proof of Proposition 3.1. We denote by $\mathscr{A}$ the space of functions of $L^2(B)$ which depend only on $\varepsilon(k)$. Note that $\mathscr{A}$ is closed in $L^2(B)$. According to Lemmas 3.2 to 3.4, the following estimate holds for any f such that $\beta < f(k) < 1 - \beta$ a.e.:

$$-E_g(f) \geqslant C_3 C_{2,\beta}\, d^2(H(f), M) + C_{1,\beta}\, d^2(H(f), \mathscr{A}) \tag{3.37}$$

where d denotes the distance associated to $L^2(B)$.

Note now that since M is finite-dimensional and since $\mathscr{A}$ is closed (in $L^2(B)$), $\mathscr{A} + M$ is also closed (in $L^2(B)$). Then, according to the open mapping theorem (see ref. 5 for example), we get a constant $C_\beta > 0$ such that for any f verifying the estimate $\beta < f(k) < 1 - \beta$ a.e.:

$$-E_g(f) \geqslant C_\beta\, d^2(H(f), M \cap \mathscr{A}) \tag{3.38}$$

Since $M \cap \mathscr{A}$ is the space of functions spanned by 1 and ε, then, according to the estimate

$$\forall x, y \in \mathbb{R}, \qquad \left| \frac{\exp x}{1 + \exp x} - \frac{\exp y}{1 + \exp y} \right| \leqslant |x - y| \tag{3.39}$$

we get

$$\begin{aligned} -E_g(f) &\geqslant C_\beta \inf_{a, c \in \mathbb{R}} \int_B |H(f) - a - c\varepsilon(k)|^2 \, dk \\ &\geqslant C_\beta \inf_{F \in \mathscr{F}_c} \int_B |f - F|^2 \, dk \end{aligned} \tag{3.40}$$

We now prove a corollary of Proposition 3.1 which concerns the scaling described in the introduction. We can prove that the scaled quantity f^α is at a distance of order α of the space of centered Fermi–Dirac functions:

Corollary 3.5. Suppose that f^α is a solution to the resealed problem (2.5)–(2.7) under Assumptions 1 to 7. Suppose moreover that it satisfies the bound (2.28). Then there exists a family of centered Fermi–Dirac functions $(F^\alpha)_{\alpha \in]0,1]}$ and a constant $C_T > 0$ such that

$$f^\alpha = F^\alpha + \alpha r^\alpha \tag{3.41}$$

with

$$\|r^\alpha\|^2_{L^2([0,T] \times \Omega \times B)} \leqslant C_T \tag{3.42}$$

Proof of Corollary 3.5. Multiplying Eq. (2.5) by $\alpha^2 H(f^\alpha)$ and integrating with respect to (t, x, k) on $[0, T] \times \Omega \times B$, we get:

$$\alpha^2 \left(\int_\Omega S^\alpha(T, x)\, dx - \int_\Omega S^\alpha(0, x)\, dx \right) + \alpha \int_0^T \int_{\partial\Omega} G^\alpha(t, x) \cdot \nu(x)\, d\lambda(x)\, dt$$

$$- \int_0^T \int_\Omega E_g(f^\alpha)\, dx\, dt - \alpha^2 \int_0^T \int_\Omega \int_B Q_1^\alpha(f^\alpha)\, H(f^\alpha)\, dk\, dx\, dt = 0 \tag{3.43}$$

In Eq. (3.43) λ denotes the superficial measure on $\partial\Omega$, S^α is the entropy defined by

$$S^\alpha(t, x) = \int_B \theta(f^\alpha(t, x, k))\, dk \tag{3.44}$$

and G^α is the entropy flux defined by

$$G^\alpha(t, x) = \int_B \nabla_k \varepsilon(k)\, \theta(f^\alpha(t, x, k))\, dk \tag{3.45}$$

In Eqs. (3.44) and (3.45), θ denotes the strictly convex function defined on $[0, 1]$ by

$$\theta(x) = x \log x + (1 - x) \log(1 - x) \tag{3.46}$$

The proof of (3.43) can be made more rigorous by first noticing that, since $\beta \leqslant f \leqslant 1 - \beta$, the function $\theta(f)$ is Lipschitz continuous with respect to f. Therefore, we can choose it to renormalize the Boltzmann equation[16] and get

$$\frac{\partial \theta(f^\alpha)}{\partial t} + \frac{1}{\alpha} \left(\nabla_k \varepsilon(k) \cdot \nabla_x + \nabla_x V \cdot \nabla_k \right) \theta(f^\alpha)$$

$$= \frac{H(f^\alpha)}{\alpha^2} \left(Q_e(f^\alpha) + Q_0(f^\alpha) \right) + H(f^\alpha)\, Q_1^\alpha(f^\alpha)$$

and we obtain (3.43) thanks to an integration over all variables (the continuity with respect to time is important).

Inserting Eq. (2.23) into Eq. (2.24) and using the evenness of ε (Assumption 1), we get:

$$1 = \int_{B_+(x)} R(k' \to k)\, \delta(\varepsilon(k) - \varepsilon(k'))\, dk' \tag{3.47}$$

Equation (3.47) means that the constant function equal to 1 satisfies the boundary condition (2.7). Hence, Jensen's inequality yields, $\forall t \in \mathbb{R}_+$, $\forall (x, k) \in \partial\Omega \times B$ such that $k \in B_-(x)$,

$$\theta(f_-^\alpha(t, x, k)) \leqslant \int_{B_+(x)} R(k' \to k)\, \delta(\varepsilon(k) - \varepsilon(k'))\, \theta(f_+^\alpha(t, x, k'))\, dk' \tag{3.48}$$

Multiplying by $|\nabla_k \varepsilon(k) \cdot \nu(x)|$, integrating with respect to $k \in B_-(x)$ and using equation (2.23) gives:

$$\int_{B_-(x)} |\nabla_k \varepsilon(k) \cdot \nu(x)|\, \theta(f^\alpha(t, x, k))\, dk \leqslant \int_{B_+(x)} |\nabla_k \varepsilon(k) \cdot \nu(x)|\, \theta(f^\alpha(t, x, k))\, dk \tag{3.49}$$

This implies that $\forall (t, x) \in \mathbb{R}_+ \times \partial\Omega$,

$$G^\alpha(t, x) \cdot \nu(x) \geqslant 0 \tag{3.50}$$

Now, according to Proposition 3.1, there exists a constant $C_\beta > 0$ and a Fermi–Dirac function $F^\alpha(t, x, k)$ such that:

$$-E_g(f^\alpha) \geqslant C_\beta(\|f^\alpha - F^\alpha\|_{L^2(B)} - \alpha^2) \tag{3.51}$$

Since $x \to x \log x + (1-x)\log(1-x)$ is a bounded function on $[0, 1]$, we deduce from (3.43) and (3.51) that

$$\|f^\alpha - F^\alpha\|^2_{L^2([0,T]\times\Omega\times B)} \leqslant C\alpha^2 + \alpha^2 \int_0^T \int_\Omega \int_B Q_1^\alpha(f^\alpha)\, H(f^\alpha)\, dk\, dx\, dt \quad (3.52)$$

Corollary 2.5 is then a straightforward consequence of Assumption 4.

4. MEAN COMPACTNESS PROPERTY

This section is aimed at proving the following result:

Proposition 4.1. Let f^α be a solution to the resealed problem (2.5)–(2.7) under Assumptions 1 to 7 satisfying the bound (2.28). Then f^α converges up to a subsequence when α tends to 0 towards a centered Fermi–Dirac function F^0 in $L^p([0,T]\times\Omega\times B)$ (strong) for $1 \leqslant p < +\infty$.

Moreover, the concentration $\rho^\alpha(t,x) = \int_B f^\alpha(t,x,k)\,dk$ and the energy $W^\alpha(t,x) = \int_B f^\alpha(t,x,k)\,\varepsilon(k)\,dk$ converge (also up to extraction) strongly in $L^p([0,T]\times\Omega)$ for $1 \leqslant p < +\infty$, when α tends to 0, respectively to $\rho^0(t,x)$ and $W^0(t,x)$, which are the concentration and energy relative to F^0.

In order to prove Proposition 4.1, we use an averaging lemma stating that ρ^α and W^α are strongly compact locally in $L^2([0,T]\times\Omega)$. Then one has to prove that the limits ρ^0 and W^0 of these quantities are indeed the concentration and energies relative to a Fermi–Dirac function F^0. Once this result is obtained, the convergence of f^α towards F^0 is a simple consequence of Corollary 3.5. The outline of the proof follows closely the previous work by Golse and Poupaud.[21] Many details are however quite different.

Lemma 4.2. Let $\tilde{f}^\alpha$, $\tilde{H}^\alpha$ be uniformly bounded in $L^2(\mathbb{R}\times\mathbb{R}^3\times B)$ and $\tilde{g}^\alpha$ be uniformly bounded in $(L^2(\mathbb{R}\times\mathbb{R}^3\times B))^3$. Suppose moreover that

$$\alpha \frac{\partial}{\partial t} \tilde{f}^\alpha + v(k)\cdot\nabla_x \tilde{f}^\alpha = \nabla_k \cdot \tilde{g}^\alpha + \tilde{H}^\alpha \quad (4.1)$$

where $k \to v(k)$ is a function of $(W^{1,\infty}(B))^3)$ satisfying the following property:

$$\exists C, \zeta > 0, \qquad \forall \omega \in S^3, \gamma > 0, \qquad \left|\left\{k \in B, \left|\begin{pmatrix} v(k) \\ 1 \end{pmatrix}\cdot\omega\right| \leqslant \gamma\right\}\right| \leqslant C\gamma^\zeta \quad (4.2)$$

Then, for any $\phi \in W^{1,\infty}(B)$, the averages $\tilde{I}^{\alpha}_{\phi}(t, x) = \int_B \tilde{f}^{\alpha}(t, x, k)\, \phi(k)\, dk$ are uniformly bounded in $L^2(\mathbb{R}_t; H^{\zeta/4}(\mathbb{R}^3_x))$.

For the proof of this lemma, we refer to ref. 21 where only the case $\phi \equiv 1$ is treated. The extension to any $\phi \in W^{1,\infty}(B)$ is straightforward.

Lemma 4.3. Let f^{α} be a solution to the resealed problem (2.5)–(2.7) under Assumptions 1 to 7 satisfying the bound (2.28). Then the concentration $\rho^{\alpha}(t, x)$ and the energy $W^{\alpha}(t, x)$ are uniformly bounded in $L^2_{\text{loc}}(\,]0, T[\,; H^{\xi/4}_{\text{loc}}(\Omega))$ (ξ is defined in Assumption 1).

Proof of Lemma 4.3. Plugging decomposition (3.41) of Corollary 3.5 in Eq. (2.5) and multiplying by α, we get:

$$\alpha \frac{\partial f^{\alpha}}{\partial t} + (\nabla_k \varepsilon(k) \cdot \nabla_x + \nabla_x V \cdot \nabla_k)\, f^{\alpha}$$
$$= (D^1 Q_e(F^{\alpha}) + Q_0)(r^{\alpha}) + \alpha (D^2 Q_e(F^{\alpha})(r^{\alpha}, r^{\alpha}) + Q_1^{\alpha}(f^{\alpha}))$$
$$+ \alpha^2 D^3 Q_e(F^{\alpha})(r^{\alpha}, r^{\alpha}, r^{\alpha}) \tag{4.3}$$

where $D^i Q_e(F^{\alpha})$ for $i = 1,..., 4$ denote respectively the ith derivative of Q_e with respect to F^{α}. (Note that since Q_e is cubic, its fourth derivative satisfies $D^4 Q_e(F^{\alpha}) = 0$). Let us now define on $\mathbb{R} \times \mathbb{R}^3 \times B$ the function $\tilde{f}^{\alpha} = \eta f^{\alpha}$, where $\eta(t, x) \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^3)$ has its support in $]0, T[\, \times \Omega$ and will be chosen later. The function $\tilde{f}^{\alpha}$ defined on $\mathbb{R} \times \mathbb{R}^3$, is a solution of the following equation:

$$\alpha \frac{\partial \tilde{f}^{\alpha}}{\partial t} + \nabla_k \varepsilon(k) \cdot \nabla_x \tilde{f}^{\alpha} = \nabla_k \cdot \tilde{g}^{\alpha} + \tilde{H}^{\alpha} \tag{4.4}$$

where

$$\tilde{g}^{\alpha} = \eta\, \nabla_x V f^{\alpha}, \qquad \tilde{H}^{\alpha} = f^{\alpha} \frac{\partial \eta}{\partial t} + f^{\alpha}\, \nabla_k \varepsilon(k) \cdot \nabla_x \eta + \eta h^{\alpha} \tag{4.5}$$

and h^{α} denotes the right-hand side of Eq. (4.3).

Note first that since $0 \leqslant f^{\alpha} \leqslant 1$, and thanks to Assumption 6, the sequences $\tilde{f}^{\alpha}$ and $|\tilde{g}^{\alpha}|$ are uniformly bounded in $L^2(\mathbb{R} \times \mathbb{R}^3 \times B)$. Moreover Assumption 1 also implies that $\nabla_k \varepsilon$ satisfies the requirements of Lemma 4.2 on v with $\zeta = \xi$.

It remains to prove that $\tilde{H}^{\alpha}$ is uniformly bounded in $L^2(\mathbb{R} \times \mathbb{R}^3 \times B)$. It is clearly enough to prove that h^{α} is uniformly bounded in $L^2([0, T] \times \Omega \times B)$. We shall therefore prove that all the terms appearing in the right-hand side of Eq (4.3) are uniformly bounded in L^2.

(I) *The term* $Q_0(r^\alpha)$. Using Assumption 3 and Cauchy–Schwarz inequality, we can prove that Q_0 is bounded in $L^2([0, T] \times \Omega \times B)$. Namely,

$$\begin{aligned}
&\|Q_0(f)\|^2_{L^2([0,T]\times\Omega\times B)} \\
&\quad\leqslant C_0^2 \int_0^T \int_\Omega \int_B \int_{\varepsilon^{-1}(\varepsilon(k))} |f' - f|^2 \frac{dN_{\varepsilon(k)}(k')}{N(\varepsilon(k))}\, dk\, dx\, dt \\
&\quad\leqslant 2C_0^2 \int_0^T \int_\Omega \int_B \int_{k' \in \varepsilon^{-1}(\varepsilon(k))} (|f'|^2 + |f|^2) \frac{dN_{\varepsilon(k)}(k')}{N(\varepsilon(k))}\, dk\, dx\, dt \\
&\quad= 4C_0^2 \int_0^T \int_\Omega \int_B \int_{k' \in \varepsilon^{-1}(\varepsilon(k))} |f|^2 \frac{dN_{\varepsilon(k)}(k')}{N(\varepsilon(k))}\, dk\, dx\, dt \\
&\quad= 4C_0^2 \|f\|^2_{L^2([0,T]\times\Omega\times B)} \qquad (4.6)
\end{aligned}$$

Then, Corollary 3.5 implies that $Q_0(r^\alpha)$ is uniformly bounded in $L^2([0, T] \times \Omega \times B)$.

(II) *The term* $D^1Q_e(F^\alpha)(r^\alpha)$. Note first that this term can be written under the form

$$\begin{aligned}
D^1Q_e(F^\alpha)(r^\alpha) = \int_{B^3} &\Phi_e\, \delta_\varepsilon\, \delta_p (r'^\alpha P^\alpha(k'_1, k, k_1) + r'^\alpha_1 P^\alpha(k', k, k_1) \\
&- r^\alpha P^\alpha(k_1, k', k'_1) - r_1^\alpha P^\alpha(k, k', k'_1))\, dk_1\, dk'\, dk'_1 \qquad (4.7)
\end{aligned}$$

where

$$P^\alpha(k_1, k', k'_1) = F_1^\alpha(1 - F'^\alpha)(1 - F'^\alpha_1) + F'^\alpha F'^\alpha_1(1 - F_1^\alpha) \qquad (4.8)$$

The function P^α is always nonnegative and bounded by 2.

We first consider the term involving r^α. According to Assumption 3 and using Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\int_0^T \int_\Omega \int_B \left| \int_{B^3} \Phi_e\, \delta_\varepsilon\, \delta_p r^\alpha P^\alpha(k_1, k', k'_1)\, dk_1\, dk'\, dk'_1 \right|^2 dk\, dx\, dt \\
&\quad\leqslant C_e^2 \int_0^T \int_\Omega \int_B |r^\alpha|^2 \int_B \sum_{g \in \mathscr{P}_{k, k_1}} \int_{\tilde{\varepsilon}^{-1}(k, k_1, g)(0)} |P^\alpha(k_1, k', k + k_1 + g - k')|^2 \\
&\qquad\times \frac{d\widetilde{N}_{k, k_1, g}(k')}{\overline{N}(k, k_1)}\, dk_1\, dk\, dx\, dt \\
&\quad\leqslant 4C_e^2\, |B|\, \|r^\alpha\|^2_{L^2([0,T]\times\Omega\times B)} \qquad (4.9)
\end{aligned}$$

According to formula (2.37), the term involving r_1^α can be treated exactly in the same way. Then, the terms involving r'^α and $r_1'^\alpha$ are treated with the help of formulas (2.37) and (2.38) and give rise to the same estimate (4.9).

(III) *The term $D^2Q_e(F^\alpha)(r^\alpha, r^\alpha)$.* We first write this term under the form

$$D^2Q_e(F^\alpha)(r^\alpha, r^\alpha) = \int_{B^3} \Phi_e\, \delta_\varepsilon\, \delta_p (r^\alpha r'^\alpha (F_1^\alpha - F_1'^\alpha) + r_1^\alpha r_1'^\alpha (F^\alpha - F'^\alpha) + r^\alpha r_1^\alpha (F'^\alpha - (1 - F_1'^\alpha)) + r'^\alpha r_1'^\alpha ((1 - F^\alpha) - F_1^\alpha) + r^\alpha r_1'^\alpha (F_1^\alpha - F'^\alpha) + r_1^\alpha r'^\alpha (F^\alpha - F_1'^\alpha))\, dk_1\, dk'\, dk_1' \tag{4.10}$$

Using the estimate $|\alpha r^\alpha| \leqslant 2$, we can find a constant $C_1 > 0$ such that:

$$\|\alpha D^2Q_e(F^\alpha)(r^\alpha, r^\alpha)\|^2_{L^2([0,T]\times\Omega\times B)} \leqslant C_1 \int_0^T \int_\Omega \int_{B^2} \sum_{g\in\mathscr{P}_{k,k_1}} \int_{k'\in\tilde{\varepsilon}^{-1}(k,k_1,g)(0)} (|r^\alpha|^2 + |r_1^\alpha|^2 + |r'^\alpha|^2 + |r_1'^\alpha|^2) \times \frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)}\, dk_1\, dk\, dx\, dt \tag{4.11}$$

Using the same symmetry properties as for D^1Q_e, we get the existence of a constant $C_2 > 0$ such that:

$$\|\alpha D^2Q_e(F^\alpha)(r^\alpha, r^\alpha)\|^2_{L^2([0,T]\times\Omega\times B)} \leqslant C_2\, \|r^\alpha\|^2_{L^2([0,T]\times\Omega\times B)} \tag{4.12}$$

(IV) *The term $D^3Q_e(F^\alpha)(r^\alpha, r^\alpha, r^\alpha)$.* Since we have

$$D^3Q_e(F^\alpha)(r^\alpha, r^\alpha, r^\alpha) = \int_{B^3} \Phi_e\, \delta_\varepsilon\, \delta_p (r^\alpha r_1^\alpha r'^\alpha + r^\alpha r_1^\alpha r_1'^\alpha - r'^\alpha r_1'^\alpha r^\alpha - r'^\alpha r_1'^\alpha r_1^\alpha)\, dk_1\, dk'\, dk_1' \tag{4.13}$$

then using once again the estimate $|\alpha r^\alpha| \leqslant 2$, we can find $C_3 > 0$ such that

$$\|\alpha^2 D^3Q_e(F^\alpha)(r^\alpha, r^\alpha, r^\alpha)\|^2_{L^2([0,T]\times\Omega\times B)} \leqslant C_3 \int_0^T \int_\Omega \int_{B^2} \sum_{g\in\mathscr{P}_{k,k_1}} \int_{k'\in\tilde{\varepsilon}^{-1}(k,k_1,g)(0)} (|r^\alpha|^2 + |r_1^\alpha|^2 + |r_1^\alpha|^2 + |r_1'^\alpha|^2) \times \frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)}\, dk_1\, dk\, dx\, dt$$

Therefore, there exists a nonnegative constant C_4 such that

$$\|\alpha^2 D^3 Q_e(F^\alpha)(r^\alpha, r^\alpha, r^\alpha)\|^2_{L^2([0,T]\times\Omega\times B)} \leqslant C_4 \|r^\alpha\|^2_{L^2([0,T]\times\Omega\times B)} \tag{4.14}$$

Note finally that because of Assumption 4, there exists a constant $K_T > 0$ such that for $\alpha \in [0,1]$,

$$\|\alpha Q_1^\alpha(f^\alpha)\|^2_{L^2([0,T]\times\Omega\times B)} \leqslant K_T \tag{4.15}$$

Then we can use Lemma 4.2 in order to prove that $\eta\rho^\alpha$ and ηW^α are uniformly bounded in $L^2(\mathbb{R}; H^{\xi/4}(\mathbb{R}^3))$. Finally, ρ^α and W^α are uniformly bounded in $L^2_{loc}(]0,T[; H^{\xi/4}_{loc}(\Omega))$.

Lemma 4.4. Assume that X_0, X and X_1 are Hilbert spaces which satisfy $X_0 \subset X \subset X_1$, with continuous inclusions. Suppose moreover that the first inclusion is compact. We denote, for any bounded set $K \subset \mathbb{R}$,

$$\mathscr{H}_K(X_0, X_1) = \{u \in L^2(\mathbb{R}, X_0),\ D_t u \in L^2(\mathbb{R}, X_1) \text{ and } \operatorname{Supp} u \subset K\} \tag{4.16}$$

where $D_t u$ denotes the derivative of u with respect to t in the sense of distributions. Then, the injection of $\mathscr{H}_K(X_0, X_1)$ into $L^2(\mathbb{R}, X)$ is compact.

For the proof of this lemma, we refer to ref. 27.

Lemma 4.5. Let f^α be a solution to the rescaled problem (2.5)–(2.7) under Assumptions 1 to 7 satisfying the bound (2.28). Then the concentration $\rho^\alpha(t,x)$ and the energy $W^\alpha(t,x)$ belong to a compact set of $L^2_{loc}(]0,T[\times\Omega)$.

Proof of Lemma 4.5. The proof is an application of Lemma 4.4. Multiplying Eq. (2.5) by $\binom{1}{\varepsilon(k)}$ and integrating with respect to k, we get:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho^\alpha \\ W^\alpha\end{pmatrix} + \frac{1}{\alpha}\int_B (\nabla_k \varepsilon \cdot \nabla_x + \nabla_x V \cdot \nabla_k)(F^\alpha + \alpha r^\alpha)\begin{pmatrix}1 \\ \varepsilon(k)\end{pmatrix} dk$$
$$= \frac{1}{\alpha^2}\int_B (Q_e + Q_0)(f^\alpha)\begin{pmatrix}1 \\ \varepsilon(k)\end{pmatrix} dk + \int_B Q_1^\alpha(f^\alpha)\begin{pmatrix}1 \\ \varepsilon(k)\end{pmatrix} dk \tag{4.17}$$

Since 1 and $\varepsilon(k)$ are collisional invariants of $Q_e + Q_0$ and since F^α and ε are even with respect to k, this identity can be rewritten under the following form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho^\alpha \\ W^\alpha\end{pmatrix} + \nabla_x \cdot \int_B r^\alpha \nabla_k \varepsilon(k)\begin{pmatrix}1 \\ \varepsilon(k)\end{pmatrix} dk - \nabla_x V \cdot \int_B r^\alpha \begin{pmatrix}0 \\ \nabla_k \varepsilon(k)\end{pmatrix} dk$$
$$= \int_B Q_1^\alpha(f^\alpha)\begin{pmatrix}1 \\ \varepsilon(k)\end{pmatrix} dk \tag{4.18}$$

Therefore, the quantities ρ^α and W^α are uniformly bounded in the space $H^1([0,T]; H^{-1}(\Omega))$. Introducing once again the cutoff function η (as in Lemma 4.3), and using Lemma 4.4 with $u=\eta\rho$, (and then $u=\eta W$), $X_0=H^{\xi/4}(\mathbb{R}^3)$, $X=L^2(\mathbb{R}^3)$ and $X_1=H^{-1}(\mathbb{R}^3)$, get Lemma 4.5.

Before turning to the proof of Proposition 4.1, we give a last lemma which specifies the link between the conservative variables (ρ, W) and the entropic variables (a, c) relative to a Fermi–Dirac function F.

Lemma 4.6. Let F be a centered Fermi–Dirac function:

$$F(k)=\frac{\exp(a+\varepsilon(k)\,c)}{1+\exp(a+\varepsilon(k)\,c)} \tag{4.19}$$

and let $\rho=\int_B F(k)\,dk$, $W=\int_B \varepsilon(k)\,F(k)\,dk$ denote its conservative variables. Then the function T defined by

$$T(a,c)=\int_B \log(1+\exp(a+\varepsilon(k)\,c))\,dk \tag{4.20}$$

belongs to $C^2(\mathbb{R}^2)$, is strictly convex and its derivatives are

$$\frac{\partial T}{\partial a}=\rho, \qquad \frac{\partial T}{\partial c}=W \tag{4.21}$$

Moreover the function E: $(a,c)\to(\rho,W)$ is a C^1-diffeomorphism from $\mathbb{R}^2$ to $E(\mathbb{R}^2)$.

Proof of Lemma 4.6. It is obvious that $T\in C^2(\mathbb{R}^2)$. The computation of its derivatives is also simple. In order to prove that T is strictly convex, we compute its Hessian matrix

$$\begin{pmatrix} \displaystyle\int_B \frac{\exp(a+\varepsilon(k)\,c)}{(1+\exp(a+\varepsilon(k)\,c))^2}\,dk & \displaystyle\int_B \frac{\exp(a+\varepsilon(k)\,c)}{(1+\exp(a+\varepsilon(k)\,c))^2}\,\varepsilon(k)\,dk \\ \displaystyle\int_B \frac{\exp(a+\varepsilon(k)\,c)}{(1+\exp(a+\varepsilon(k)\,c))^2}\,\varepsilon(k)\,dk & \displaystyle\int_B \frac{\exp(a+\varepsilon(k)\,c)}{(1+\exp(a+\varepsilon(k)\,c))^2}\,\varepsilon^2(k)\,dk \end{pmatrix} \tag{4.22}$$

According to Cauchy–Schwarz inequality and using the linear independence of 1 and ε, it becomes clear that T is strictly convex. We note that the Jacobian matrix of E is nothing but the Hessian matrix of T. Then the properties of E are a straightforward application of the inverse function theorem.

We now can prove Proposition 4.1.

Proof of Proposition 4.1. According to Lemma 4.5, the sequences ρ^α and W^α admit a subsequence $\rho^{\alpha_1^n}$ and $W^{\alpha_1^n}$ converging for a.e. $(t, x) \in [0, T] \times \Omega$ towards a limit ρ^0 and W^0.

Note also that because of Corollary 3.5, we can find a subsequence α^n of α_1^n such that for a.e. (t, x, k) in $[0, T] \times \Omega \times B$, $f^{\alpha^n}(t, x, k) - F^{\alpha^n}(t, x, k)$ tends to 0. Then the conservative variables $\rho_F^{\alpha^n}$ and $W_F^{\alpha^n}$, which are related to the Fermi–Dirac functions F^{α^n}, also converge, for a.e. $(t, x) \in [0, T] \times \Omega$ towards ρ^0 and W^0.

Let us prove that for a.e. $(t_0, x_0) \in [0, T] \times \Omega$ the entropic variables a^{α^n} and c^{α^n} related to the Fermi–Dirac function F^{α^n} are bounded. To this aim, we introduce for $(t_0, x_0) \in [0, T] \times \Omega$ the set $\mathscr{L}_{t_0, x_0} = \{k \in B, f^{\alpha^n}(t_0, x_0, k) - F^{\alpha^n}(t_0, x_0, k) \to 0$ and $\beta \leqslant f^{\alpha^n}(t_0, x_0, k) \leqslant 1 - \beta\}$, and the set $\mathscr{M} = \{(t_0, x_0) \in [0, T] \times \Omega,\ |\mathscr{L}^c_{t_0, x_0}| = 0\}$. Then, $\mathscr{M}$ is a set of full measure of $[0, T] \times \Omega$.

Assume that $a^{\alpha^n}(t_0, x_0)$ is unbounded, then there exists a subsequence α_2^n such that

$$\lim_{n \to +\infty} |a^{\alpha_2^n}(t_0, x_0)| = +\infty \tag{4.23}$$

Then, for all $k \in \mathscr{L}_{t_0, x_0}$ such that $\varepsilon(k) \neq 0$ and

$$\lim_{n \to +\infty} \frac{c^{\alpha_2^n}}{a^{\alpha_2^n}}(t_0, x_0) \neq -\frac{1}{\varepsilon(k)} \tag{4.24}$$

(when this limit exists), the sequence

$$a^{\alpha_2^n}(t_0, x_0) + \varepsilon(k)\, c^{\alpha_2^n}(t_0, x_0) = a^{\alpha_2^n}(t_0, x_0) \left(1 + \varepsilon(k) \frac{c^{\alpha_2^n}}{a^{\alpha_2^n}}(t_0, x_0)\right) \tag{4.25}$$

is unbounded, and therefore $H(F^{\alpha_2^n}(t_0, x_0, k))$ and $H(f^{\alpha_2^n}(t_0, x_0, k))$ are also unbounded. But this is impossible since for a.e. $k \in B$, $\beta \leqslant f^{\alpha^n}(t_0, x_0, k) \leqslant 1 - \beta$. Hence $a^{\alpha^n}(t_0, x_0)$ is bounded. The same argument shows that $c^{\alpha^n}(t_0, x_0)$ is bounded. Consequently, there exists $R_{t_0, x_0} > 0$ such that $\forall n \in \mathbb{N}$,

$$\begin{pmatrix} a^{\alpha^n} \\ c^{\alpha^n} \end{pmatrix} (t_0, x_0) \in \bar{B}(R_{t_0, x_0}) \tag{4.26}$$

It means that

$$\begin{pmatrix} \rho_F^{\alpha^n} \\ W_F^{\alpha^n} \end{pmatrix} (t_0, x_0) \in E(\bar{B}(0, R)) \tag{4.27}$$

and therefore,

$$\begin{pmatrix} \rho^0 \\ W^0 \end{pmatrix}(t_0, x_0) \in E(\bar{B}(0, R)) \tag{4.28}$$

Then, since E^{-1} is continuous on $E(\mathbb{R}^2)$ (see Lemma 4.6), we have:

$$\begin{pmatrix} a^{\alpha^n} \\ c^{\alpha^n} \end{pmatrix}(t_0, x_0) \to \begin{pmatrix} a^0 \\ c^0 \end{pmatrix}(t_0, x_0) = E^{-1}\left(\begin{pmatrix} \rho^0 \\ W^0 \end{pmatrix}(t_0, x_0)\right) \tag{4.29}$$

This in turn implies that for a.e. $k \in B$,

$$F^{\alpha^n}(t_0, x_0, k) = \frac{\exp(a^{\alpha^n}(t_0, x_0) + \varepsilon(k)\, c^{\alpha^n}(t_0, x_0))}{1 + \exp(a^{\alpha^n}(t_0, x_0) + \varepsilon(k)\, c^{\alpha^n}(t_0, x_0))} \to$$

$$F^0(t_0, x_0, k) = \frac{\exp(a^0(t_0, x_0) + \varepsilon(k)\, c^0(t_0, x_0))}{1 + \exp(a^0(t_0, x_0) + \varepsilon(k)\, c^0(t_0, x_0))} \tag{4.30}$$

Then, f^{α^n} also converges a.e. towards the Fermi–Dirac function F^0. The convergence in L^p (strong) for all $1 \leqslant p < +\infty$ of f^{α^n} and its moments is then a consequence of its uniform boundedness.

5. CONVERGENCE TO THE ENERGY TRANSPORT MODEL

We conclude in this section the proof of Theorem 1.

Proof of Theorem 1. According to Propositions 3.1 and 4.1, the sequence f^α gives rise to a subsequence f^{α^n} converging in L^p (for $1 \leqslant p < +\infty$) towards a centered Fermi–Dirac function F^0. Moreover, according to Corollary 3.5 one can extract another subsequence (simply denoted by α in the sequel) such that r^α converges weakly in L^2 towards a limit r^0. Let us now prove prove that formulae (2.30), (2.31) and (2.35) hold.

Multiplying Eq. (25) by $\binom{1}{\varepsilon(k)}$ and integrating with respect to k, we get (see the Proof of Lemma 4.5),

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho^\alpha \\ W^\alpha \end{pmatrix} + \nabla_x \cdot \int_B r^\alpha\, \nabla_k \varepsilon(k) \begin{pmatrix} 1 \\ \varepsilon(k) \end{pmatrix} dk - \nabla_x V \cdot \int_B r^\alpha \begin{pmatrix} 0 \\ \nabla_k \varepsilon(k) \end{pmatrix} dk$$

$$= \int_B Q_1^\alpha(f^\alpha) \begin{pmatrix} 1 \\ \varepsilon(k) \end{pmatrix} dk \tag{5.1}$$

Passing to the limit in the sense of distributions in Eq. (5.1), we get

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho^0\\ W^0\end{pmatrix}+\nabla_x\cdot\int_B r^0\,\nabla_k\varepsilon(k)\begin{pmatrix}1\\ \varepsilon(k)\end{pmatrix}dk-\nabla_x V\cdot\int_B r^0\begin{pmatrix}0\\ \nabla_k\varepsilon(k)\end{pmatrix}dk$$
$$=\begin{pmatrix}0\\ \int_B Q_1^0(F^0)\,\varepsilon(k)\,dk\end{pmatrix}\tag{5.2}$$

which proves (2.30) and (2.31).

Besides, Eq. (2.5) can be put under the form

$$\alpha\frac{\partial f^\alpha}{\partial t}+(\nabla_k\varepsilon(k)\cdot\nabla_x+\nabla_x V\cdot\nabla_k)\,f^\alpha$$
$$=(D^1Q_e(F^\alpha)+Q_0)(r^\alpha)+\alpha D^2Q_e(F^\alpha)(r^\alpha,r^\alpha)$$
$$+\alpha^2D^3Q_e(F^\alpha)(r^\alpha,r^\alpha,r^\alpha)+\alpha Q_1^\alpha(f^\alpha)\tag{5.3}$$

To pass to the limit $\alpha\to 0$, we first notice that

$$\alpha\frac{\partial f^\alpha}{\partial t}+(\nabla_k\varepsilon(k)\cdot\nabla_x+\nabla_x V\cdot\nabla_k)\,f^\alpha\rightharpoonup(\nabla_k\varepsilon(k)\cdot\nabla_x+\nabla_x V\cdot\nabla_k)\,F^0\tag{5.4}$$

in the sense of distributions. We now pass to the limit in the right hand side of Eq. (5.3) (also in the sense of distributions). It is clear that $Q_0(r^\alpha)$ tends to $Q_0(r^0)$ because Q_0 is a linear bounded operator of $L^2([0,T]\times\Omega\times B)$ (see the proof of Lemma 4.3). Besides, $D^2Q_e(F^\alpha)(r^\alpha,r^\alpha)$ is bounded in $L^1([0,T]\times\Omega\times B)$. Indeed, since

$$D^2Q_e(F^\alpha)(r^\alpha,r^\alpha)=\int_{B^3}\Phi_e\,\delta_\varepsilon\,\delta_p\{r^\alpha r'^\alpha(F_1^\alpha-F_1'^\alpha)+r_1^\alpha r_1'^\alpha(F^\alpha-F'^\alpha)$$
$$+r^\alpha r_1^\alpha(F'^\alpha-(1-F_1'^\alpha))+r'^\alpha r_1'^\alpha((1-F^\alpha)-F_1^\alpha)$$
$$+r^\alpha r_1'^\alpha(F_1^\alpha-F'^\alpha)+r_1^\alpha r'^\alpha(F^\alpha-{}'^{\alpha}_1)\}\,dk_1\,dk'\,dk_1'\tag{5.5}$$

the estimate $0\leqslant F^\alpha\leqslant 1$ implies the existence of a constant $C_1>0$ such that:

$$\|D^2Q_e(F^\alpha)(r^\alpha,r^\alpha)\|_{L^1([0,T]\times\Omega\times B)}$$
$$\leqslant C_1\sum_{g\in\mathscr{P}_{k,k_1}}\int_0^T\int_\Omega\int_{B^2}\int_{k'\in\tilde{\varepsilon}^{-1}(k,k_1,g)(0)}(|r^\alpha|^2+|r_1^\alpha|^2+|r'^\alpha|^2+|r_1'^\alpha|^2)$$
$$\times\frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)}\,dk_1\,dk\,dx\,dt\tag{5.6}$$

Using the symmetry properties (2.36), (2.37), we get the existence of a constant $C_2 > 0$ such that:

$$\|D^2 Q_e(F^\alpha)(r^\alpha, r^\alpha)\|^2_{L^2([0,T]\times\Omega\times B)} \leqslant C_2 \, \|r^\alpha\|^2_{L^2([0,T]\times\Omega\times B)} \tag{5.7}$$

It is also clear (because $|\alpha r^\alpha| \leqslant 2$) that the term

$$D^3 Q_e(F^\alpha)(r^\alpha, r^\alpha, r^\alpha)$$

$$= \int_{B^3} \Phi_e \, \delta_\varepsilon \, \delta_p (r^\alpha r_1^\alpha r'^\alpha + r^\alpha r_1^\alpha r_1'^\alpha - r'^\alpha r_1'^\alpha r^\alpha - r'^\alpha r_1'^\alpha r_1^\alpha) \, dk_1 \, dk' \, dk_1'$$

satisfies the estimate

$$\|\alpha D^3 Q_e(F^\alpha)(r^\alpha, r^\alpha, r^\alpha)\|_{L^1([0,T]\times\Omega\times B)} \leqslant C_3 \, \|r^\alpha\|^2_{L^2([0,T]\times\Omega\times B)} \tag{5.8}$$

for some constant $C_3 > 0$.

It remains to prove that $D^1 Q_e(F^\alpha)(r^\alpha)$ converges weakly in $L^1([0,T] \times \Omega \times B)$ towards $D^1 Q_e(F^0)(r^0)$. We remark that $D^1 Q_e(F^0)$ is a bounded linear operator of $L^1([0,T] \times \Omega \times B)$. Namely, using the notations of the proof of Lemma 4.3,

$$\|D^1 Q_e(F^0)(r^\alpha)\|_{L^1([0,T]\times\Omega\times B)}$$

$$= \int_0^T \int_\Omega \int_B dk \, dx \, dt \, \Bigg| \int_B dk_1 \sum_{g \in \mathscr{P}_{k,k_1}} \int_{\tilde{\varepsilon}^{-1}(k,k_1,g)(0)} \bar{N}(k,k_1)$$

$$\times \Phi_e \{ r'^\alpha P^0(k+k_1+g-k', k, k_1) + r^\alpha(k+k_1+g-k') \, P^0(k', k, k_1)$$

$$- r^\alpha P^0(k_1, k', k+k_1+g-k') - r_1^\alpha P^0(k, k', k+k_1+g-k') \} \, \frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)} \Bigg|$$

$$\leqslant 2C_e \int_0^T \int_\Omega \int_B \int_B \sum_{g \in \mathscr{P}_{k,k_1}} \int_{\tilde{\varepsilon}^{-1}(k,k_1,g)(0)} (|r^\alpha| + |r^{\alpha'}| + |r_1^\alpha| + |r_1^{\alpha'}|)$$

$$\times \frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)} \, dk_1 \, dk \, dx \, dt$$

Therefore, we have the following estimate

$$\|D^1 Q_e(F^0)(r^\alpha)\|_{L^1([0,T]\times\Omega\times B)} \leqslant 8C_e \, |B| \, \|r^\alpha\|_{L^1([0,T]\times\Omega\times B)} \tag{5.9}$$

which implies that $D^1 Q_e(F^0)(r^\alpha)$ converges towards $D^1 Q_e(F^0)(r^0)$ in $L^1([0,T] \times \Omega \times B)$ weak. It remains to prove that

$$D^1 Q_e(F^\alpha)(r^\alpha) - D^1 Q_e(F^0)(r^\alpha) \to 0 \tag{5.10}$$

in $L^1([0, T] \times \Omega \times B)$ (strong). With the notations of Lemma 4.3, we have

$$\begin{aligned}
&\|D^1 Q_e(F^\alpha)(r^\alpha) - D^1 Q_e(F^0)(r^\alpha)\|_{L^1([0,T]\times\Omega\times B)} \\
&\quad \leqslant \int_0^T \int_\Omega \int_B dk\, dx\, dt \left| \int_B \sum_{g \in \mathscr{P}_{k,k_1}} \int_{\tilde{\varepsilon}^{-1}(k,k_1,g)(0)} \bar{N}\Phi_e \right. \\
&\qquad \times \{ r'^\alpha (P^\alpha(k_1', k, k_1) - P^0(k_1', k, k_1)) \\
&\qquad + r_1'^\alpha (P^\alpha(k', k, k_1) - P^0(k', k, k_1)) - r^\alpha (P^\alpha(k_1, k', k_1') - P^0(k_1, k', k_1')) \\
&\qquad \left. - r_1^\alpha (P^\alpha(k, k', k_1') - P^0(k, k', k_1'))\} \frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)}\, dk_1 \right|
\end{aligned} \tag{5.11}$$

Using the boundedness of $\Phi_e \bar{N}$, the right hand side of this inequality can be estimated by $\sqrt{I}\,\sqrt{II}$ where

$$\begin{aligned}
I = &\int_0^T \int_\Omega \int_B \int_B \sum_{g \in \mathscr{P}_{k,k_1}} \int_{\tilde{\varepsilon}^{-1}(k,k_1,g)(0)} (|r^\alpha|^2 + |r^{\alpha'}|^2 + |r_1^\alpha|^2 + |r_1^{\alpha'}|^2) \\
&\times \frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)}\, dk_1\, dk\, dx\, dt
\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}
II = &\int_0^T \int_\Omega \int_B \int_B \sum_{g \in \mathscr{P}_{k,k_1}} \int_{\tilde{\varepsilon}^{-1}(k,k_1,g)(0)} (|P^\alpha(k_1', k, k_1) - P^0(k_1', k, k_1)|^2 \\
&+ |P^\alpha(k', k, k_1) - P^0(k', k, k_1)|^2 + |P^\alpha(k_1, k', k_1') - P^0(k_1, k', k_1')|^2 \\
&+ |P^\alpha(k, k', k_1') - P^0(k, k', k_1')|^2) \frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)}\, dk_1\, dk\, dx\, dt
\end{aligned} \tag{5.13}$$

In view of (2.37), it is easy to show that

$$I \leqslant 4\,|B|\,\|r_\alpha\|^2_{L^2([0,T]\times\Omega\times B)} \tag{5.14}$$

whereas

$$\begin{aligned}
II \leqslant &\int_0^T \int_\Omega \int_B \int_B \sum_{g \in \mathscr{P}_{k,k_1}} \int_{\tilde{\varepsilon}^{-1}(k,k_1,g)(0)} |P^\alpha(k', k, k_1) - P^0(k', k, k_1)|^2 \\
&\times \frac{d\tilde{N}_{k,k_1,g}(k')}{\bar{N}(k,k_1)}\, dk_1\, dk\, dx\, dt
\end{aligned} \tag{5.15}$$

From (4.8), we get

$$
\begin{aligned}
II \leqslant & \int_0^T \int_\Omega \int_B \int_B \sum_{g \in \mathscr{P}_{k, k_1}} dk_1 \, dk \, dx \, dt \\
& \times \int_{\tilde{\varepsilon}^{-1}(k, k_1, g)(0)} \{ |F_1^\alpha - F_1^0|^2 + |F_1^\alpha|^2 \, |F'^\alpha - F'^0|^2 \\
& + |F'^0|^2 \, |F_1^\alpha - F_1^0| + |F_1^\alpha|^2 \, |F_1'^\alpha - F_1'^0|^2 + |F_1'^0|^2 \, |F_1^\alpha - F_1^0|^2 \\
& + |F'^\alpha|^2 \, |F_1'^\alpha - F_1'^0|_2 + |F_1'^0|^2 \, |F'^\alpha - F'^0|^2 \} \, \frac{d\tilde{N}_{k, k_1, g}(k')}{\bar{N}(k, k_1)} \\
\leqslant & \, 7 \, |B| \, \| F^\alpha - F^0 \|_{L^2([0, T] \times \Omega \times B)}
\end{aligned}
\tag{5.16}
$$

Therefore, in view of all the above estimates, we can to the limit in (5.3) and prove (2.35). The only thing left to show is that $J^0 \cdot \nu$ and $J_W^0 \cdot \nu$ vanish on the boundary $\partial\Omega$. This is a direct consequence of mass and energy conservation of the reflection operator. Indeed, for a distribution function satisfying the boundary condition (2.7), we have

$$
\int_B \nabla\varepsilon(k) \cdot \nu(x) \, G(\varepsilon(k)) \, f \, dk = 0 \qquad \forall x \in \partial\Omega
$$

Consequently since $f^\alpha = F^\alpha + \alpha r^\alpha$ where F^α is a centered Fermi–Dirac distribution, and therefore even with respect to k, we have for all $x \in \partial\Omega$,

$$
\int_B r^\alpha \varepsilon(k) \, \nabla\varepsilon(k) \cdot \nu(x) \, dk = \int_B r^\alpha \, \nabla\varepsilon(k) \cdot \nu(x) \, dk = 0
$$

which in the limit $\alpha \to 0$ gives $J^0 \cdot \nu = J_W^0 \cdot \nu = 0$. This proof can be made more rigorous by taking test functions and passing to the limit in the weak formulation of the Boltzmann equation (2.26). Indeed, the test functions $\theta(x, p, t) = \psi(x, t)$ and $\theta(x, k, t) = \varepsilon(k) \, \psi(x, t)$ are such that $\mathscr{B}(f^\alpha, \theta) = 0$ (see (2.27)). We can then pass to the weak limit in (2.26) and get

$$
\begin{aligned}
& \int_\Omega \psi(x, 0) \int_B f_{in} \begin{pmatrix} 1 \\ \varepsilon(k) \end{pmatrix} dk \, dx - \int_{\mathbb{R}^+} \int_\Omega \frac{\partial\psi}{\partial t} \begin{pmatrix} \rho^0 \\ W^0 \end{pmatrix} dx \, dt \\
& \quad - \int_{\mathbb{R}^+} \int_\Omega \nabla_x \psi \cdot \int_B r^0 \, \nabla_k \varepsilon(k) \begin{pmatrix} 1 \\ \varepsilon(k) \end{pmatrix} dk \, dx \, dt \\
& \quad - \int_{\mathbb{R}^+} \int_\Omega \psi(x, t) \, \nabla_x V \cdot \int_B r^0 \begin{pmatrix} 0 \\ \nabla_k \varepsilon(k) \end{pmatrix} dk \, dx \, dt \\
& = \int_{\mathbb{R}^+} \int_\Omega \psi(x, t) \begin{pmatrix} 0 \\ \int_B Q_1^0(F^0) \, \varepsilon(k) \, dk \end{pmatrix} dx \, dt
\end{aligned}
$$

which is exactly the weak formulation of the Energy transport model with the boundary and initial conditions announced in Theorem 1.

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REFERENCES

1. Y. Apanovich, E. Lyumskis, B. Polsky, A. Shur, and P. Blakey, Steady-state and transient analysis of submicron devices using energy-balance and simplified hydrodynamic models, *IEEE Trans. CAD of Integ. Circuits Syst.* **13**:702–711 (1994).
2. C. Bardos, F. Golse, B. Perthame, and R. Sentis, The nonaccretive radiative transfer equations: Existence of solutions and Rosseland approximation, *J. Funct. Anal.* **77**(2):434–460 (1988).
3. N. Ben Abdallah and P. Degond, On a hierarchy of macroscopic models for semiconductors, *J. Math. Phys.* **37**(7):3306–3333 (1996).
4. N. Ben Abdallah, P. Degond, and S. Genieys, An energy transport model for semiconductors derived from the Boltzmann equation, *J. Stat. Phys.* **84**(1–2):205–231 (1996).
5. H. Brezis, *Analyse Fonctionnelle, Théorie et Applications* (Masson, Paris, 1983).
6. C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer, Berlin, 1988).
7. C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases* (Springer Verlag, New York, 1994).
8. D. Chen, E. Kan, U. Ravaioli, C. Shu, and R. Dutton, An improved energy transport model including nonparabolicity and non-maxwellian distribution effects, *IEEE Electr. Dev. Letters* **13**:26–28 (1992).
9. P. Degond, S. Génieys, and A. Jüngel, An existence and uniqueness result for the energy transport model in semiconductor theory, *C. R. Acad. Sci. Sér. I* **324**:29–34 (1997).
10. P. Degond, S. Génieys, and A. Jüngel, An existence result for strongly coupled parabolic system arising in non equilibrium thermodynamics, *C. R. Acad. Sci. Sér. I* **325**:227–232 (1997).
11. P. Degond, S. Génieys, and A. Jüngel, A steady-state system in non equilibrium thermodynamics including thermal and electrical effects, *Math. Meth. Appl. Sci.* **21**:1399–1413 (1998).
12. P. Degond, S. Génieys, and A. Jüngel, A system of parabolic equations in non equilibrium thermodynamics including thermal and electrical effects, *J. Math. Pures Appl.* **76**:991–1015 (1997).
13. P. Degond, A. Jüngel, and P. Pietra, Numerical discretization of energy-transport models for semiconductors with non-parabolic band structure, article in preparation.
14. P. Degond and C. Schmeiser, Macroscopic models for semiconductor heterostructures, *J. Math. Phys.* **39**(9):4634–4663 (1998).
15. L. Desvillettes, Entropy dissipation rate and convergence in kinetic equations, *Comm. Math. Phys.* **123**:687–702 (1989).

16. R. Di Perna and P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* **98**(3):511–547 (1989).
17. J. Dolbeault, Kinetic models and quantum effects: A modified Boltzmann equation for Fermi–Dirac particles, *Arch. Rat. Mech. Anal.* **127**(2):101–131 (1994).
18. H. Federer, *Geometric Measure Theory* (Springer-Verlag, Berlin, 1969).
19. S. Génieys, Energy transport model for a nondegenerate semiconductor. Convergence of the Filbert expansion in the linearized case, *Asympt. Anal.* **17**(4):279–308 (1998).
20. F. Golse, P. L. Lions, B. Perthame, and R. Sentis, Regularity of the moments of the solution of a transport equation, *J. Funct. Anal.* **77**:110–125 (1988).
21. F. Golse and F. Poupaud, Limite fluide des equations de Boltzmann des semiconducteurs pour une statistique de Fermi-Dirac, *Asympt. Anal.* **6**:135–160 (1992).
22. K. Hamdache, Initial boundary value problems for Boltzmann equation. Global existence of weak solutions, *Arch. Rat. Mech. Anal.* **119**:309–353 (1992).
23. A. Marrocco, P. Montarnal, and B. Perthame, Simulation of the energy-transport and simplified hydrodynamic models for semiconductor devices using mixed finite elements, Proceedings ECCOMAS 96, London, 1996. John Wiley.
24. C. Schmeiser and A. Zwirchmayr, Convergence of moment methods for linear kinetic equations, *SIAM J. Numer. Anal.* **36**(1):74–88 (1999).
25. K. Souissi, F. Odeh, H. Tang, and A. Gnudi, Comparative studies of hydrodynamic and energy transport models, *COMPEL* **13**:439–453 (1994).
26. R. Stratton, Diffusion of hot and cold electrons in semiconductor barriers, *Phys. Rev.* **126**:2002–2014 (1962).
27. R. Temam, *Navier-Stokes Equations*, Studies in Mathematics and its Applications (North-Holland, 1984).
28. B. Wennberg, On an entropy dissipation inequality for the Boltzmann equation, *C. R. Acad. Sci. Sér. I* **315**(13):1441–1446 (1992).